数学者と学ぶ量子力学

SG2022-10

1 Hilbert space

Definition (Banach space)
A normed space $B = (B, \|\cdot\|)$ is called a Banach space, if B is complete with respect to $\|\cdot\|$.

Definition (Hilbert space)
An inner product space $H = (H, \langle\cdot,\cdot\rangle)$ is called a Hilbert space, if H is complete with respect to the induced norm $\|\cdot\|$.

Definition (bounded linear operator)
Let B_1 and B_2 are Banach spaces. An linear operator (map) L is called bounded, if

$$\|L\|_{\mathrm{op}} := \sup_{x \in B_1\setminus\{0\}} \frac{\|Lx\|_{B_2}}{\|x\|_{B_1}} < \infty.$$

Theorem (Riesz represent theorem)
For any bounded linear operator $f : H \to \mathbb{C}$, there exists a unique $\xi \in H$ such that $f(\eta) = \langle \xi, \eta\rangle$ for all $\eta \in H$.

2 C^*-algebra

Definition (Banach algebra)
A Banach algebra $\mathcal{A}$ is an algebra and also a Banach space such that $\|ab\| \le \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

Definition (C^*-algebra)
A Banach algebra $\mathcal{A}$ with involution $*$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$, is called a C^*-algebra. In particular, $\mathcal{A}$ has a unit I, we call $\mathcal{A}$ a unital C^*-algebra.

Definition (representation)
A representation of $\mathcal{A}$ on H is a linear map $\pi : \mathcal{A} \to B(H)$ such that $\pi(a^*) = \pi(a)^*$ and $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in \mathcal{A}$.

Definition (positive linear functional)
A linear operator $\varphi : \mathcal{A} \to \mathbb{C}$ is called positive, if $\varphi(a^*a) \ge 0$ for all $a \in \mathcal{A}$.

Definition (state)
A positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called a state, if $\|\varphi\| = 1$.

Denote the total set of states on $\mathcal{A}$ by $S(\mathcal{A})$.

Theorem (GNS construction)
Let $\mathcal{A}$ be a unital C^*-algebra and $\omega \in S(\mathcal{A})$. Then there exists a pair of a representation π, a Hilbert space H and $\Omega \in H$ such that $\omega(a) = \langle \Omega, \pi(a)\Omega\rangle$ for all $a \in \mathcal{A}$.

3 Quasi-local algebra

Let Γ be a countable set, $\mathcal{P}(\Gamma)$ be the collection of subsets of Γ,

$$\mathcal{P}_{\mathrm{f}}(\Gamma) := \{\Lambda \in \mathcal{P}(\Gamma) |\ |\Lambda| := \#\Lambda < \infty\}.$$

We assume that at each site $x \in \Gamma$ there is a d-dimensional quantum spin system with observable (operator) algebra $\mathcal{A}(\{x\}) := M_d(\mathbb{C})$. For $\Lambda \in \mathcal{P}_{\mathrm{f}}(\Gamma)$, we define the corresponding algebra $\mathcal{A}(\Lambda)$ of observables by

$$\mathcal{A}(\Lambda) := \bigotimes_{x\in\Lambda} \mathcal{A}(\{x\}) = \bigotimes_{x\in\Lambda} M_d(\mathbb{C}).$$

Since for $\Lambda_1 \subset \Lambda_2$, $C^*(\Lambda_1)$ is embedded in $C^*(\Lambda_2)$ by the mapping

$$C^*(\Lambda_1) \ni a \mapsto a \otimes I_{\mathcal{A}(\Lambda_2\setminus\Lambda_1)} \in C^*(\Lambda_2),$$

we are able to define the C^*-algebra $\mathcal{A}_{\mathrm{loc}}(\Gamma)$ by

$$\mathcal{A}_{\mathrm{loc}}(\Gamma) := \bigcup_{\Lambda\in\mathcal{P}_{\mathrm{f}}(\Gamma)} \mathcal{A}(\Lambda)$$

with the norm defined by $\|x\| := \|x\|_{\mathcal{A}(\Lambda)}$ for $x \in \mathcal{A}(\Lambda)$ and $\Lambda \in \mathcal{P}_{\mathrm{f}}(\Gamma)$.

Definition (quasi-local algebra)
We define the quasi-local algebra $\mathcal{A}(\Gamma)$ by

$$\mathcal{A}(\Gamma) := \overline{\mathcal{A}_{\mathrm{loc}}(\Gamma)}^{\|\cdot\|},$$

where $\overline{\cdot}^{\|\cdot\|}$ is the completion with respect to the norm $\|\cdot\|$ on $\mathcal{A}_{\mathrm{loc}}(\Gamma)$.

Definition (closed two-sided ideal)
A closed subspace I of a C^*-algebra $\mathcal{A}$ is called a closed two-sided ideal (or simply ideal), if $ab, ba \in I$ for all $a \in \mathcal{A}$ and $b \in I$.

Definition (simple)
A C^*-algebra $\mathcal{A}$ is called simple, if any closed two-sided ideal of $\mathcal{A}$ equals to $\mathcal{A}$ or $\{0\}$.

Proposition
Any quasi-local algebra $\mathcal{A}(\mathbb{Z}^d)$ for a quantum spin system is simple.

数学者と学ぶ量子力学

SG2022-10

4 Hamiltonian

Definition (interaction)
Let $\mathcal{A}$ be a quasi-local algebra of a quantum spin system on Γ. An interaction Φ on Γ is defined by a map $\Phi : \mathcal{P}_f(\Gamma) \to \mathcal{A}$ such that for each $\Lambda \in \mathcal{P}_f(\Gamma)$, $\Phi(\Lambda)$ is selfadjoint and belongs to $\mathcal{A}(\Lambda)$.

An interaction Φ on Γ is called bounded, if

$$\|\Phi\| := \sup_{x\in\Gamma} \sum_{\Lambda\in\mathcal{P}_f(\Gamma);x\in\Lambda} \|\Phi(\Lambda)\|.$$

An interaction Φ on Γ is called finite range, if there exists $c > 0$ such that $\Phi(\Lambda) = 0$ whenever $\sup_{x,y\in\Lambda} d(x,y) > c$.

Definition (local Hamiltonian)
For $\Lambda \in \mathcal{P}_f(\Gamma)$ the local Hamiltonian H_Λ of interaction Φ on Γ is defined by

$$H_\Lambda := \sum_{X\subset\Lambda} \Phi(X), \quad \Lambda \in \mathcal{P}_f(\Gamma).$$

Definition (derivation)
A derivation δ of a C^*-algebra $\mathcal{A}$ is a linear map from a $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ such that

- $\delta(A^*) = \delta(A)^*$ for all $A \in D(\delta)$.
- $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in D(\delta)$.

For a bounded finite range interaction Φ,

$$\delta(A) := \lim_{\Lambda\uparrow\Gamma} \mathrm{i}[H_\Lambda, A]$$

defines a derivation with domain $D(\delta) = \mathcal{A}_{\mathrm{loc}}$.

Lemma
Let Φ be a bounded finite range interaction on Γ and δ be a corresponding derivation. Then,

$$\alpha_t(A) := \exp(t\delta)(A) := \sum_{k=0}^{\infty} \frac{t^n}{k!}\delta^n(A), \quad A \in \mathcal{A}(\Gamma)$$

defines a strongly continuous group $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ of automorphism on $\mathcal{A}(\Gamma)$.

Definition (C^*-dynamical system)
A C^*-dynamical system $(\mathcal{A}, \alpha)$ is a pair of a C^*-algebra and a strongly continuous group $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ of automorphism on $\mathcal{A}$.

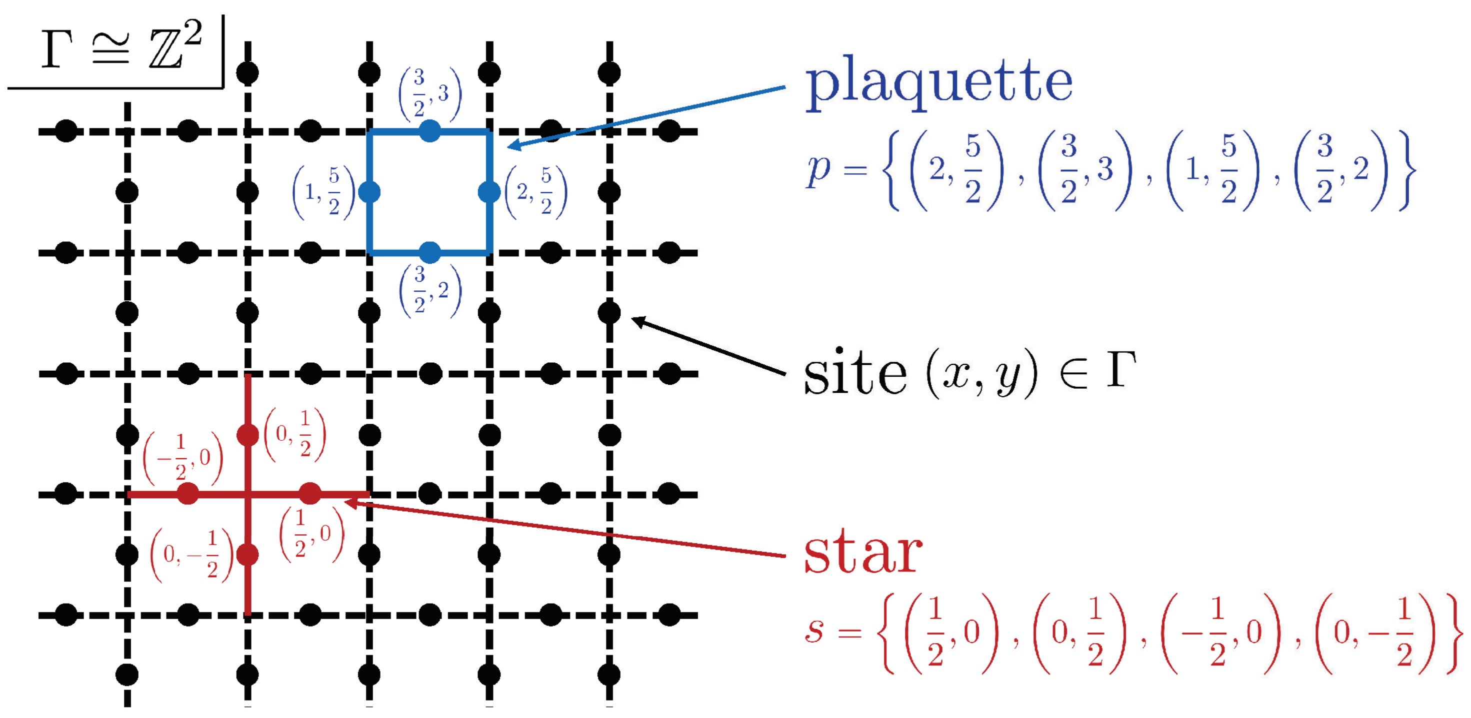
Definition (ground state)
Let $(\mathcal{A}, \alpha)$ is a C^*-dynamical system. An state $\omega \in \mathfrak{A}$ is called a α-ground state, if $\omega \circ \alpha_t = \omega$ for all $t \in \mathbb{R}$ and the representation H_ω appeared in GNS construction is positive.

5 Toric code

To introduce Toric code, let

$$\Gamma := \{\text{edges between nearest neighbor points in } \mathbb{Z}^2\},$$
$$\mathcal{A}(\{x\}) := M_2(\mathbb{C}), \quad x \in \Gamma,$$
$$\sigma_j^x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_j^z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

From $\mathcal{A}(\{x\})$ we define the quasi-local algebra $\mathcal{A}(\Gamma)$.



A toric code is a model with an interaction Φ given by

$$\Phi(\Lambda) := \begin{cases} -A_s, & \Lambda = s \text{ for some star } s, \\ -B_p, & \Lambda = p \text{ for some plaquette } p, \\ 0, & \text{else,} \end{cases}$$

where $A_s := \bigotimes_{j\in s} \sigma_j^x$ (star operator)

$B_p := \bigotimes_{j\in p} \sigma_j^z$ (plaquette operator).

Remark
In the case that $\Gamma := \{\text{edges of } (\mathbb{Z}/N\mathbb{Z})^2\}$ (finite system), the ground state ω_0 is given by

$$\omega_0 := \prod_{s:\text{star}} \frac{I + A_s}{\sqrt{2}} |\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle$$

where $|\uparrow\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For all star s and plaquette p,

- $[A_s, B_p] := A_sB_p - B_pA_s = 0$.
- $A_s^2 = B_p^2 = I$.
- $I - A_s$, $I - B_p$ are positive.

Proposition
If ω is a state on $\mathcal{A}(\Gamma)$ such that $\omega(A_s) = \omega(B_p) = 1$ for all star s and plaquette p, then ω is a ground state.

Theorem
Toric code has a translation invariant grand state ω_0.

Reference

[1] Pieter Naaijkens, Quantum Spin System on Infinite Lattice, Lecture Notes in Physics 933, Springer Cham, arXiv 1311.2717v2.